

On an arithmetic inequality on $\mathbb{P}_{\mathbb{Q}}^1$

Mounir Hajli*

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Abstract

We establish an inequality comparing the height and the χ -arithmetic volume of toric metrized divisors on $\mathbb{P}_{\mathbb{Q}}^1$. This gives a partial answer to a question of Burgos, Moriwaki, Philippon and Sombra ([5, remark 5.13]).

In [5, remark 5.13] the authors ask if the following inequality

$$h_{\overline{D}}(X) \leq \widehat{\text{vol}}_{\chi}(X, \overline{D})$$

holds for any toric DSP metrized \mathbb{R} -divisor \overline{D} on $X = \mathbb{P}_{\mathbb{K}}^1$, where \mathbb{K} is a global field, $h_{\overline{D}}(X)$ is the height of X and $\widehat{\text{vol}}_{\chi}(X, \overline{D})$ is χ -arithmetic volume with respect to \overline{D} .

In this note we give an affirmative answer to this question when $\mathbb{K} = \mathbb{Q}$, D is nef and \overline{D} is a toric DSP metrized divisor such that the metric on all non-archimedean places is the canonical metric (see theorem (0.2)).

Let L be a line bundle on $\mathbb{P}^1(\mathbb{C})$. A metric $\|\cdot\|$ on L is semipositive if it is the uniform limit of a sequence of semipositive smooth metrics. The metric $\|\cdot\|$ is DSP if it is the quotient of two semipositives ones. We denote by $\mathcal{M}_{\mathbb{Q}}$ the set of places of \mathbb{Q} . For any $v \in \mathcal{M}_{\mathbb{Q}}$, we denote by $\mathbb{P}_v^{1,an}$ the v -adic analytification of $\mathbb{P}_{\mathbb{Q}}^1$. Similarly a line bundle L on $\mathbb{P}_{\mathbb{Q}}^1$ defines a collection of analytic line bundles $\{L_v^{an}\}_{v \in \mathcal{M}_{\mathbb{Q}}}$, see [5, §3] for more details.

Definition 0.1. A metrized divisor on $\mathbb{P}_{\mathbb{Q}}^1$ is a pair $\overline{D} = (D, (\|\cdot\|_v)_{v \in \mathcal{M}_{\mathbb{Q}}})$ formed by a divisor D with $\|\cdot\|_{\infty}$ is a continuous hermitian metric on $\mathcal{O}(D)_{\infty}$ and $\|\cdot\|_v$ is the canonical metric of $\mathcal{O}(D)_v$ for v a non-archimedean place. We say that \overline{D} is smooth or semipositive if so is the metric $\|\cdot\|_{\infty}$. We say that \overline{D} is a DSP divisor if it is the difference of two semipositive divisors. The Green function of \overline{D} is the function $g_{\overline{D}} : \mathbb{P}^1(\mathbb{C}) \setminus |D| \rightarrow \mathbb{R}$ given by

$$g_{\overline{D}}(p) = -\log \|s_D(p)\|_{\infty},$$

where s_D is the canonical section of $\mathcal{O}(D)$.

Let \overline{D} be a metrized DSP divisor on $\mathbb{P}_{\mathbb{Q}}^1$ as in (0.1). We suppose that \overline{D} is toric, see [5, §.4]. This means that D is a toric divisor and $\|\cdot\|_{\infty}$ is invariant under the action of \mathbb{S}^1 the compact torus of $\mathbb{P}^1(\mathbb{C})$ (see [5, definition 4.12] and [5, proposition 4.16]). In the sequel, we assume that \overline{D} satisfies these hypothesis and D is nef.

*National Center for Theoretical Sciences (Taipei Office) National Taiwan University, Taipei 106, Taiwan
E-mail: hajlimounir@gmail.com

Theorem 0.2. *Under the previous hypothesis, we have*

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) \leq \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}).$$

In order to prove this theorem, we assume first that \overline{D} is smooth. By definition, $g_{\overline{D}}$ is a smooth weight of $\|\cdot\|_{\infty}$. We denote by $Pg_{\overline{D}}$ the equilibrium weight of $g_{\overline{D}}$ (see the appendix) instead of $P_{\mathbb{P}^1}g_{\overline{D}}$ and by $\|\cdot\|_P$ the hermitian metric defined by $Pg_{\overline{D}}$ and we denote by \overline{D}_P the metrized divisor D endowed with the metric $\|\cdot\|_P$ on the archimedean place and with the canonical metric on all non-archimedean places.

Claim 0.3. *\overline{D}_P is a semipositive toric divisor.*

Proof. By definition $Pg_{\overline{D}}$ is a psh weight on $\mathcal{O}(D)_{\infty}$ and we know that $\|\cdot\|_P$ is a continuous metric (see for instance [2, §1.4, before (1.8)]). Then the Chern current $c_1((\mathcal{O}(D), P\|\cdot\|))$ is semipositive¹. By [6, theorem 4.6.1]², $\|\cdot\|_P$ is a semipositive metric.

Let g be a psh weight function on $\mathcal{O}(D)_{\infty}$ with $g \leq g_{\overline{D}}$. Let $\theta \in \mathbb{S}^1$. We set g_{θ} the function given by $g_{\theta}(z) = g(\theta \cdot z)$ for any $z \in \mathbb{P}^1(\mathbb{C})$. Then g_{θ} is clearly a psh weight on $\mathcal{O}(D)_{\infty}$. We have $g_{\theta}(z) = g(\theta \cdot z) \leq g_{\overline{D}}(\theta \cdot z) = g_{\overline{D}}(z)$, $\forall z \in \mathbb{P}^1(\mathbb{C})$. Then, $g_{\theta}(z) \leq Pg_{\overline{D}}(z)$, $\forall z \in \mathbb{P}^1(\mathbb{C})$. Therefore, $Pg_{\overline{D}}(\theta \cdot z) \leq Pg_{\overline{D}}(z)$, $\forall \theta \in \mathbb{S}^1, \forall z \in \mathbb{P}^1(\mathbb{C})$. We conclude that

$$Pg_{\overline{D}}(\theta \cdot z) = Pg_{\overline{D}}(z) \quad \forall \theta \in \mathbb{S}^1, \forall z \in \mathbb{P}^1(\mathbb{C}).$$

Which means that $\|\cdot\|_P$ is an invariant metric. We conclude that \overline{D}_P is a semipositive toric divisor on $\mathbb{P}_{\mathbb{Q}}^1$. \square

Recall that if $\overline{D}' := (D, (\|\cdot\|'_v)_{v \in \mathcal{M}_{\mathbb{Q}}})$ is a smooth metrized divisor as in (0.1), then by [4, proposition 3.2.2], we have

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) - h_{\overline{D}'}(\mathbb{P}_{\mathbb{Q}}^1) = - \int_X (g_{\overline{D}} - g_{\overline{D}'}) (c_1(\mathcal{O}(D), \|\cdot\|) + c_1(\mathcal{O}(D), \|\cdot\|')).$$

By [7], one can extend this equality to the case of DSP divisor \overline{D}' , and we have

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) - h_{\overline{D}'}(\mathbb{P}_{\mathbb{Q}}^1) = - \int_X (g_{\overline{D}} - g_{\overline{D}'}) (c_1(\mathcal{O}(D), \|\cdot\|) + c_1(\mathcal{O}(D), \|\cdot\|')),$$

where $c_1(\mathcal{O}(D), \|\cdot\|')$ is the first Chern current of $(\mathcal{O}(D), \|\cdot\|')$.

Since \overline{D}_P is semipositive, then the previous equality gives

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) - h_{\overline{D}_P}(\mathbb{P}_{\mathbb{Q}}^1) = - \int_X (g_{\overline{D}} - g_{\overline{D}_P}) (c_1(\mathcal{O}(D), \|\cdot\|) + c_1(\mathcal{O}(D), \|\cdot\|_P)).$$

From (6), we have

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) \leq h_{\overline{D}_P}(\mathbb{P}_{\mathbb{Q}}^1). \tag{1}$$

Since \overline{D}_P is a semipositive toric divisor, then by [5, corollary 5.8]

$$h_{\overline{D}_P}(\mathbb{P}_{\mathbb{Q}}^1) = \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}_P), \tag{2}$$

and by [5, theorem 5.6], we have

$$\widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}_P) = 2 \int_{\Delta_D} \vartheta_{\overline{D}_P} d\text{vol}_{\mathbb{R}},$$

where $\vartheta_{\overline{D}_P}$ is the roof function associated to \overline{D}_P (see [5, definition 4.17]).

¹that is $c_1((\mathcal{O}(D), P_X\|\cdot\|)) \geq 0$

²Notice that a semipositive metric as in (0.1) corresponds to the notion of admissible metric in [7] and in [6, 4.5.5]

Claim 0.4. *We have,*

$$\vartheta_{\overline{D}_P} = \vartheta_{\overline{D}},$$

on Δ_D .

Proof. This is an easy consequence of the combination of [5, proposition 5.1 (1)] and [3, proposition 2.8]. Indeed, by [3, proposition 2.8] we have $\sup_{\mathbb{P}^1} \|s\|_{k\overline{D}} = \sup_{\mathbb{P}^1} \|s\|_{k\overline{D}_P}$ for any $s \in H^0(\mathbb{P}^1, \mathcal{O}(kD))$ and $k \in \mathbb{N}^*$. But, we know that $\sup_{\mathbb{P}^1} \|s_m\| = \exp(-k\vartheta_{\overline{D}}(\frac{m}{k}))$ where s_m is the global section of $\mathcal{O}(kD)$ corresponding to $m \in k\Delta_D \cap \mathbb{Z}$ (see for instance [5, proposition 5.1]). By continuity and density arguments we deduce the equality of the claim. \square

By [5, theorem 5.6] and the claim (0.4) we have,

$$\widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}_P) = \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}). \quad (3)$$

Then from (1), (2) and (3) we conclude that

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) \leq \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}).$$

Thus we prove the theorem (0.2) for \overline{D} smooth. Now let \overline{D} be a toric DSP divisor. By definition, there exist $\overline{D}_1 = (D_1, \|\cdot\|_1)$ and $\overline{D}_2 = (D_2, \|\cdot\|_2)$ two semipositive toric divisors such that $D = D_1 - D_2$ and $\|\cdot\| = \|\cdot\|_1 \otimes \|\cdot\|_2^{-1}$. For $i = 1, 2$, we choose $(\|\cdot\|_{i,n})_{n \in \mathbb{N}}$ a sequence of smooth and semipositives metrics³ on \overline{D}_i converging uniformly to $\|\cdot\|_i$. We set $\|\cdot\|_n := \|\cdot\|_{1,n} \otimes \|\cdot\|_{2,n}^{-1}$ and $\overline{D}_n := (D, \|\cdot\|_n)$ for any $n \in \mathbb{N}$. This is a sequence of smooth metrics on $\mathcal{O}(D)$ converging uniformly to $\|\cdot\|$. The smooth case implies that

$$h_{\overline{D}_n}(\mathbb{P}_{\mathbb{Q}}^1) \leq \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}_n) \quad \forall n \in \mathbb{N}. \quad (4)$$

By [7], we have that the LHS of (4) converges to $h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1)$. Moreover, we can establish that the roof functions of a sequence of metrics converging uniformly form a sequence of continuous functions on Δ_D converging uniformly. We deduce that the RHS of (4) converges to $\int_{\Delta_D} \vartheta_{\overline{D}} d\text{vol}_{\mathbb{R}}$ that is to $\widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D})$, by [5, theorem 5.6]. We conclude that

$$h_{\overline{D}}(\mathbb{P}_{\mathbb{Q}}^1) \leq \widehat{\text{vol}}_{\chi}(\mathbb{P}_{\mathbb{Q}}^1, \overline{D}).$$

This ends the proof of the theorem (0.2).

1 Appendix

Let X be compact manifold of dimension n and L an ample holomorphic line bundle on X . Let ϕ be a weight of a continuous hermitian metric $e^{-\phi}$ on L . When ϕ is smooth we define the Monge-Ampère operator as

$$\text{MA}(\phi) := (dd^c \phi)^{\wedge n}.$$

The equilibrium weight of ϕ is defined as:

$$P_X \phi := \sup^* \{ \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}$$

where $*$ denotes upper semicontinuous regularization. When ϕ is smooth then $P_X \phi = \sup \{ \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}$. It is known that $P_X \phi$ is a psh weight and the metric $e^{-P_X \phi}$ is continuous (see for instance [2, §1.4, before (1.8)]). By the theory of Bedford-Taylor, the Monge-Ampère operator can be extended to

³Here semipositive, means that the associated first Chern form is semipositive

locally bounded psh weights ϕ (see [1]).

By [3, proposition 2.10] we have

$$\int_X (P_X \phi - \phi) \text{MA}(P_X \phi) = 0. \quad (5)$$

When $\dim(X) = 1$, we have

$$\int_X (\phi - P_X \phi) (\text{MA}(\phi) + \text{MA}(P_X \phi)) \leq 0 \quad (6)$$

Indeed,

$$\begin{aligned} \frac{1}{2} \int_X (\phi - P_X \phi) (\text{MA}(\phi) + \text{MA}(P_X \phi)) &= \frac{1}{2} \int_X (\phi - P_X \phi) (dd^c \phi - dd^c P_X \phi) \quad \text{by (5)} \\ &= - \int_X d(\phi - P_X \phi) \wedge d^c(\phi - P_X \phi) \\ &\leq 0. \end{aligned}$$

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